TOPOLOGY - III, SOLUTION SHEET 10

Exercise 1. (1) Let N_1 and N_2 be neighbourhoods of x_0 and y_0 in X and Y respectively, which deformation retract onto x_0 and y_0. The $N := N_1 \cup N_2$ is a contractible open subset of the wedge sum $X \vee Y$. Let $U := X \cup N$ and $V := Y \cup N$, then we have the Mayer Vietoris long exact sequence in reduced homology:

$$... \to \tilde{H}_n(N) \to \tilde{H}_n(U) \oplus \tilde{H}_n(V) \to \tilde{H}_n(X \vee Y) \to \tilde{H}_{n-1}(N) \to ...$$

Now note that $\tilde{H}_k(N) = 0$ for all k and U, V deformation retract onto X, Y respectively. So the exercise follows from the Mayer-Vietoris sequence.

(2) Let S_g denote the wedge sum of g copies of S^1. Using the first part of the exercise and inducting on g, it follows that $H_1(S_g) = H_1(S^1)^{\oplus g} = \mathbb{Z}^g$ and $H_k(S_g) = 0$ for $k > 0$. Finally $H_0(S_g) = \mathbb{Z}$ since S_g is a path-connected space.

Exercise 2. Consider the planar diagram for $(T^2)^{\# n}$ given by $\Sigma = a_1 b_1 a_1^{-1} b_1^{-1} ... a_n b_n a_n^{-1} b_n^{-1}$. Let $q : \Sigma \to (T^2)^{\# n}$ be the quotient map and note that q is a homeomorphism on $\Sigma - \partial\Sigma$ and that $q(\partial\Sigma)$ is the wedge-sum of $2n$ copies of S_1. Let $V_1, V_2 \subset \Sigma$ be open subsets, where V_1 is given by Σ punctured at the centre and V_2 is a small open disk around the centre. Let $U_1 := q(V_1)$ and $U_2 := q(V_2)$ be opens covering $(T^2)^{\# n}$. Observe that U_2 is contractible and that U_1 deformation retracts onto $q(\partial\Sigma)$ since V_1 deformation retracts onto $\partial\Sigma$. Moreover $U_1 \cap U_2 \cong V_1 \cap V_2$ is a punctured open disk and deformation retracts onto a circle. We have the Mayer Vietoris sequence in reduced homology for V_1, V_2 :

$$0 \to H_2((T^2)^{\# n}) \to H_1(U_1 \cap U_2) \xrightarrow{\alpha} H_1(U_1) \oplus H_1(U_2) \xrightarrow{\beta} H_1((T^2)^{\# n}) \to \tilde{H}_0(U_1 \cap U_2) = 0.$$

This gives us the exact sequence:

$$0 \to H_2((T^2)^{\# n}) \to H_1(S^1) \xrightarrow{\alpha} H_1(\bigvee_{2n} S^1) \xrightarrow{\beta} H_1((T^2)^{\# n}) \to 0.$$

Now we show that α is the 0 map which will show that $H_2((T^2)^{\# n}) \cong H_1(S^1) = \mathbb{Z}$ and that $H_1((T^2)^{\# n}) \cong H_1(\bigvee_{2n} S^1) = \mathbb{Z}^{2n}$. Indeed α send 1 to the cycle in $H_1(U_1) \cong H_1(\bigvee_{2n} S^1)$ given by $a_1 b_1 a_1^{-1} b_1^{-1} ... a_n b_n a_n^{-1} b_n^{-1}$. One can check that this cycle is homologous to $a_1 + b_1 - a_1 - b_1 + ... + a_n + b_n - a_n - b_n = 0$. Where by representing a cycle as a word w just means a map $\Delta^1 \to (T^2)^{\# n})$ with image w.

The case for $(\mathbb{RP}^2)^{\# n}$ is done similarly and leave it to the reader to fill in the details for that case.

Exercise 3. Consider the suspension SX as a union of open subsets C_+ and C_- containing the usual copies of CX as in exercise 4 of sheet 6. Then C_+ and C_- are contractible and $C_+ \cap C_-$ deformation retracts onto X. Applying the Mayer-Vietoris long exact sequence gives us the desired result.

Exercise 4. We proceed by induction on n. The case $n = 1$ is clear. Now we apply Mayer Vietoris in reduced homology to the cover $V_1 := U_1 \cup ... \cup U_{n-1}$ and $V_2 := U_n$:

$$... \to \tilde{H}_k(U_1 \cup ... \cup U_{n-1}) \oplus \tilde{H}_k(U_n) \to \tilde{H}_k(X) \to \tilde{H}_{k-1}((U_n \cap U_1) \cup (U_n \cap U_2) \cup ... \cup (U_n \cap U_{n-1})) \to ...$$

Then applying the induction hypothesis to the spaces $U_1 \cup ... \cup U_{n-1}$ and $(U_n \cap U_1) \cup (U_n \cap U_2) \cup ... \cup (U_n \cap U_{n-1})$ for $k \geq n-1$ yields the required vanishing of $H_k(X)$.

Exercise 5. Let D be a small cylindrical open neighbourhood of the knot $K \cong S^1$ in $\mathbb{R}^3$. We apply Mayer Vietoris in reduced homology to the opens D and $\mathbb{R}^3 \backslash K$ of $\mathbb{R}^3$. Note that D deformation retracts to S_1 and $D \cap (\mathbb{R}^3 \backslash K)$ deformation retracts to a to torus T^2;

$$... \to \tilde{H}_n(T^2) \to \tilde{H}_n(S^1) \oplus \tilde{H}_n(\mathbb{R}^3 \backslash K) \to \tilde{H}_n(\mathbb{R}^3) \to \tilde{H}_{n-1}(T^2) \to ...$$

The above long exact sequence yields $H_2(\mathbb{R}^3 \backslash K) \cong \mathbb{Z}$, $H_1(\mathbb{R}^3 \backslash K) \cong \mathbb{Z}$, $H_0(\mathbb{R}^3 \backslash K) \cong \mathbb{Z}$ and $H_k(\mathbb{R}^3 \backslash K) = 0$ for $k > 2$.

Exercise 6. We refer the reader to the long exact sequence (*) in example 2.48 on page 151 of Hatcher's book. Setting f, g to be the identity map, we have the long exact sequence

$$... \xrightarrow{0} H_n(X) \xrightarrow{i_*} H_n(X \times S^1) \to H_{n-1}(X) \xrightarrow{0} H_{n-1}(X) \to ...$$

The short exact sequence

$$0 \to H_n(X) \xrightarrow{i_*} H_n(X \times S^1) \to H_{n-1}(X) \to 0$$

splits since if $p : X \times S^1 \to X$ is the projection map then $p_* : H_n(X \times S^1) \to H_n(X)$ is a section which gives a splitting on the left.